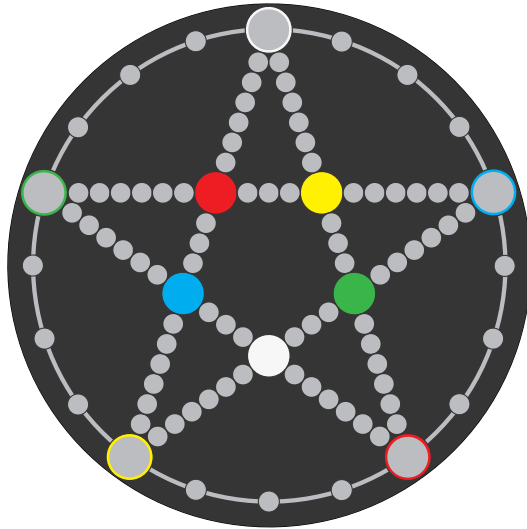


# PENTAGAME

Compendium



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# Overview

**P**ENTAGAME is a game with a simple structure, very little material and easy rules. But behind its simplicity lurks complexity: there are more possible games than stars in the sky. And while the image of the board is a simple geometric shape, it nevertheless has a rich history. This book deals with all this, and with more. It grew in parallel to the development of the game and hence is testament to it.

**The first part** explains the game as it is today, an extensive form of the rule sheets, if you like.

**The second part** explains how the idea was born and how we (re-)created Pentagame.

**The third part** shows that there might have been a pentagram shaped board before, and investigates why it was presumably lost.

**The fourth part** is an in-depth analysis of the game at hand, starting from general observations, and moving from the opening theory via strategy and tactics all the way to end game theory.

**The fifth part** deals with multi player setups and possible extensions.

**The sixth part** deals with tournaments, and how to score and compare games in general.

**The seventh part** dives into topics of advanced game theory and complexity theory. This part serves to proof some important qualities: complexity, but also drama, of the game.

**Part eight** is nothing else than a conclusion: we have a game that has all it takes to become a classic, and probably the reconstruction of an ancient game.

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The book ends with an **Annex**: A bibliography, a list of figures, a list of tables, and index, and a detailed list of content.

What this book cannot possibly convey is how and how much we enjoy playing this game. Having played hundreds of times, Pentagame continues to puzzle. Every single match was, is, and will be a rich, unique, and personal experience.

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**Part VIII.**

**Conclusion**

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# 1. Summary

We can sum up the results from all prior parts and and conclude:

1. With Pentagame we have found a proper candidate for the missing pentagonal classic, q.e.f:
  - a) We have created a pentagram shaped game board;
  - b) We have found simple rules for complex and decisive play;
2. We have sought potential ancestors:
  - a) We can explain its absence;
  - b) We find it probable that such a game has existed before *Pente grammai* (or even *Petteia*);
3. We have analysed Pentagame;
  - a) We have given advice on strategy;
  - b) We have clarified the rules such as scoring etc.,
  - c) We have proven that Pentagame has complexity, clarity, drama and decisiveness.

We thus conclude:

Pentagame is

- *probably* a resuscitated antique board game;
- *certainly* a game that has what it takes to be a classic.

q.e.d.



## 2. Outlook

The analysis of the game has shown some interesting properties, but it was a theoretical exercise. What is missing and what should be further challenges is the following:

1. A computer implementation of Pentagame to allow spatially independent play and data collection:
  - a) A data structure to record moves, and states of the board,
  - b) An interface to exchange such moves,
  - c) A graphical representation.
2. The collection of data (through such a computer implementation) to allow statistical analyses, such as:
  - a) The duration of play: average, variance etc.,
  - b) The statistical quality of (opening) moves,
  - c) The exact values of the game's qualities such as ABF, drama etc. over the course of play.
3. The development of artificial intelligence or heuristic methods to solve the game, with questions such as:
  - a) The evaluation of positions and moves,
  - b) The evaluation of (opponent's) strategies,
  - c) The best move given a situation and an estimation of the opponent's actions.
4. Study of the psychology of players, such as
  - a) The steepness of player's learning curves,
  - b) Coalition behaviour in multi-player settings.

But most of all: *the game must be played!* For whenever we play it, we fathom a cornucopia of possibilities, combinations, and ideas. It is, after all, and ought to be, a *game*.

# Annex

# Bibliography

- [1] *Ex Libro De Nymphis, Sylvanis, Pygmaeis, Salamandris, & Gigantibus &c: Item von der Massa auss welcher der Mensch geschaffen worden.* excudebat Ioannes Cruciger, 1566.
- [2] *Superior Beings: If They Exist, How Would We Know?* Springer New York, 2013.
- [3] Alcaeus. Fr. 351 Lobel / Page.
- [4] L. V. Allis, H. J. van den Herik, and I. S. Herschberg. Which games will survive? *Heuristic Programming in Artificial Intelligence 2: The Second Computer Olympiad*, pages 232–243, 1990.
- [5] L.V. Allis. *Searching for Solutions in Games and Artificial Intelligence.* Ponsen & Looijen, 1994.
- [6] Archimedes. *The Sand Reckoner.*
- [7] M. Aurelius and C. Gill. *Meditations.* Number Bücher 1-6 in Clarendon Later Ancient Philosophers. OUP Oxford, 2013.
- [8] R. G. Austin. Greek board-games. *Antiquity*, 14(55):257–271, 1940.
- [9] J.L. Benson. *Greek Color Theory and the Four Elements: A Cosmological Interpretation.* University of Massachusetts Amherst Libraries, 2000.
- [10] E.R. Berlekamp, J.H. Conway, and R.K. Guy. *Winning Ways for Your Mathematical Plays.* Number Bd. 2. CRC Press, 2018.
- [11] J. Bewersdorff. *Glück, Logik und Bluff: Mathematik im Spiel: Methoden, Ergebnisse und Grenzen.* Vieweg + Teubner, 2010.

- 
- [12] M. Borroff. *Sir Gavain and the Green Knight: A Stylistic and Metrical Study*. Yale studies in English. Yale University Press, 1962.
- [13] R. Caillois. *Les jeux et les hommes: le masque et le virtige*. Idées (Gallimard). Gallimard, 1958.
- [14] C. Calcagnini. *Caelii Calcagnini ... Opera aliquot ... per Hier. Frobenium et Nic. Episcopium*, 1544.
- [15] Cassiodor. *Institutiones*. II, 4.2.
- [16] William C. Charron. Greeks and games: Forerunners of modern game theory. *Forum for Social Economics*, 29(2):1–32, Mar 2000.
- [17] K.S. Clair. *The Secret Lives of Colour*. John Murray Press, 2016.
- [18] D. Cram and J.L. Forgeng. *Francis Willughby's Book of Games: A Seventeenth-Century Treatise on Sports, Games and Pastimes*. Taylor & Francis, 2017.
- [19] A. Crowley and Ordo Templi Orientis. *Liber O : vel manus et sagittae : sub figura VI*. Ordo Templi Orientis, 1994.
- [20] S. Cuomo. *Ancient Mathematics*. Sciences of Antiquity. Taylor & Francis, 2005.
- [21] Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitrou. The complexity of computing a Nash equilibrium. *Communications of the ACM*, 52(2), 2009.
- [22] L.B. de Fouquières. *Les jeux des anciens: leur description, leur origine, leurs rapports avec la religion, l'histoire, les arts et les moeurs*. Didier et c, 1869.
- [23] D. de Souter. *Danielis Soveri ... Palamedes sivae de tabula lusoria, alea, et variis ludis, libri tres: quorum I philogicus, II historicus, III ethicus, seu moralis*. 1622.

- [24] R. Descartes and G. Heffernan. *Discours de la Methode: Discourse on the Method: A Bilingual Edition with an Interpretive Essay*. University of Notre Dame Press, 1994.
- [25] D. Diderot and J.L.R. Alembert. *Encyclopédie Ou Dictionnaire Raisonné Des Sciences, Des Arts Et Des Métiers*. Encyclopédie Ou Dictionnaire Raisonné Des Sciences, Des Arts Et Des Métiers. Briasson, 1754.
- [26] Empedocles. *De natura*.
- [27] F. Engels. *Herrn Eugen Dührings Umwälzung der Wissenschaft: (Anti-Dühring)*. Bücherei des Marxismus-Leninismus. Dietz, 1960.
- [28] Eustathios. *In Od.*
- [29] Robert Fagen. *Animal Play Behavior*. Oxford University Press, 1981.
- [30] R. Fitzpatrick and J.L. Heiberg. *Euclid's Elements*. Richard Fitzpatrick, 2007.
- [31] D.H. Fowler. *The Mathematics of Plato's Academy: A New Reconstruction*. Clarendon Press, 1999.
- [32] Aviezri S. Fraenkel and Richard J. Nowakowski. Combinatorial games: selected short bibliography with a succinct gourmet introduction. *Games of No Chance*, 4:309–338, 2015.
- [33] P. Fraundorffer and J.A. Mercklin. *Tabula smaragdina medico-pharmaceutica.*. Zieger, 1713.
- [34] J.G. Frazer. *The Golden Bough*. BookRix, 2019.
- [35] D. Fudenberg, J. Tirole, J.A. TIROLE, and MIT Press. *Game Theory*. Mit Press. MIT Press, 1991.

- 
- [36] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. Mathematical Sciences Series. W. H. Freeman, 1979.
- [37] A. Gregory. *Plato's Philosophy of Science*. Bloomsbury Publishing, 2015.
- [38] D. Harmening. *Superstitio: Überlieferungs- und theoriegeschichtliche Untersuchungen zur kirchlich-theologischen Aberglaubensliteratur des Mittelalters*. Number Bd. 1 in *Superstitio: überlieferungs- und theoriegeschichtliche Untersuchungen zur kirchlich-theologischen Aberglaubensliteratur des Mittelalters*. E. Schmidt, 1979.
- [39] R. Herz-Fischler. *A Mathematical History of the Golden Number*. Dover Books on Mathematics. Dover Publications, 2013.
- [40] H. Hesse. *Das Glasperlenspiel: Versuch einer Lebensbeschreibung des Magister Ludi Josef Knecht samt Knechts hinterlassenen Schriften*. Suhrkamp Verlag, 2012.
- [41] E. Hoffmann, R. Klibansky, and Heidelberger Akademie der Wissenschaften. *Dialogus de ludo globi*.. Nicolai de Cusa Opera omnia. Meiner, F, 1932.
- [42] R. Hübner. Der Begriff ἀζυξ beim Brettspiel. In *...vor dem Papyrus sind alle gleich!*, pages 80–92. S. P. R. Eberhard H. Kockelmann and M. Schentuleit. Papyrologische Beiträge zu Ehren von Bärbel Kramer, 2009.
- [43] Johan Huizinga. *Homo Ludens : proeve ener bepaling van het spelelement der cultuur*. Tjeenk Willink, Groningen, 7. dr. edition, 1974.
- [44] Iamblichos. *Vita Pythagorae*.
- [45] C.L. Joost-Gaugier. *Measuring Heaven: Pythagoras and His Influence on Thought and Art in Antiquity and the Middle Ages*. Cornell University Press, 2018.

- [46] C. Kaplan. Introductory tiling theory for computer graphics, 2009.
- [47] J. Kerschensteiner. *Kosmos: Quellenkritische Untersuchungen zu den Vorsokratikern*. Zetemata. Monographien zur Klassischen Altertumswissenschaft. C.H. Beck'sche Verlagsbuchhandlung, 1962.
- [48] Stephen Kidd. Pente grammai and the 'holy line'. *Board Game Studies Journal*, 11(1):83–99, 2017.
- [49] Donald E. Knuth and Ronald W. Moore. An analysis of  $\alpha$ - $\beta$ -pruning. *Artificial Intelligence*, 6(4):293–326, 1975.
- [50] Leslie Kurke. Ancient greek games and how to play them. *Classical Philology*, 94, 1999.
- [51] Diogenes Laertius. *Vitae Philosophorum*.
- [52] Hans Lamer. Lusoria tabula. In *Paulys Real-encyclopädie der classischen Altertumswissenschaft*. 1927.
- [53] É. Lévi and E. Maturi. *Dogme et rituel de la haute magie par Éliphas Lévi: Dogme. 1*. Number Vol. 1 and 2. G: Bailliére, 1861.
- [54] C. LEVI-STRAUSS. *La pensée sauvage*. Fonds Claude Lévi-Strauss. Place des éditeurs, 2014.
- [55] J.M. Lhôte. *Histoire des jeux de société: géométries du désir*. Flammarion, 1994.
- [56] R.N. Le and R. Nölle. *Francis Bacon, Instauratio Magna. Novum Organum (lat./dt.)*. Books on Demand, 2008.
- [57] Lucian. *Pro lapsu inter salutandem*.
- [58] R. MacMullen. *Christianizing the Roman Empire: (A.D. 100-400)*. Number Teile 100-400 in A.D. 100-400. Yale University Press, 1984.

- [59] J. Martineau. *A Little Book of Coincidence: In the Solar System*. Wooden Books, 2018.
- [60] A.A. Martinez. *The Cult of Pythagoras: Math and Myths*. University of Pittsburgh Press, 2013.
- [61] P. Melanchthon. *Liber de Anima*. Mullerus, 1595.
- [62] P.G. Morencos. *Libro de Ajedrez, Dados y Tablas de Alfonso X el Sabio: Estudio*. Publicaciones del Patrimonio Nacional. Patrimonio Nacional, 1977.
- [63] H.J.R. Murray. *A History of Board-games Other Than Chess*. Hacker Art Books, 1978.
- [64] H.J.R. Murray. *A History of Chess*. Skyhorse Publishing, 2015.
- [65] John Napier. *Mirifici logarithmorum canonis descriptio ejusque usus in utraque trigonometria etc.* 1614.
- [66] T.J.F. Nash. *Essays on Game Theory*. Edward Elgar Publishing, Incorporated, 1996.
- [67] J. Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. CreateSpace Independent Publishing Platform, 2018.
- [68] S. Olsen. *The Golden Section: Nature's Greatest Secret*. Wooden Books, 2018.
- [69] Scott Olson. The indefinite dyad and the golden section: Uncovering plato's second principle. *Nexus Network Journal*, 4(1), 2002.
- [70] F.L. Pacioli. *Divina Proportione*. Kraus, 1969.
- [71] D.S. Parlett. *The Oxford History of Board Games*. Oxford University Press, 1999.



- [72] J. Pearl. *Causality*. Cambridge University Press, 2009.
- [73] Philostratus. *Vita sophistes*.
- [74] J. Pollux and E. Bethe. *Pollucis Onomasticon*. Number Teil 3 in Lexicographi Graeci. Teubner, 1967.
- [75] O. Preußler. *Krabat*. Die 50 Besten. Thienemann, 2008. (1971).
- [76] Ptolemy. *Almagest*.
- [77] E. Sachs. *Die Fünf Platonischen Körper, Zur Geschichte Der Mathematik Und Der Elementenlehre Platons Und Der Pythagoreer*. Creative Media Partners, LLC, 2018.
- [78] Ulrich Schädler. Pente grammai—the ancient greek board game five lines. pages 173–96, 2008.
- [79] H.G. Senger, P. Wilpert, N. von Kues, and H.G. Senger. *Die belehrte Unwissenheit (De docta ignorantia) / De docta ignorantia. Die belehrte Unwissenheit: Buch I. Die belehrte Unwissenheit (De docta ignorantia) / De docta ignorantia. Die belehrte Unwissenheit*. Meiner, F, 1994. (1440).
- [80] St John Simpson. Homo ludens. the earliest board games in the near east. In I.L. Finkel, editor, *Ancient Board Games in perspective: Papers from the 1990 British Museum colloquium, with additional contributions*. 2007.
- [81] J.M. Smith. *Evolution and the Theory of Games*. Cambridge University Press, 1982.
- [82] B. Snell. *Fragmente: Griechisch - deutsch*, 2014.
- [83] Sophocles, H. Lloyd-Jones, F. Storr, P.H.J. Lloyd-Jones, and F.R.P.G.S.H. Lloyd-Jones. *Sophocles: Fragments*, 1994.

- 
- [84] H. Sproemberg. *Arno Borst, Die Katharer. Stuttgart: Hiersemann 1953. XI, 372 S. 8° (Schriften der Monumenta Germaniae Historica. 12.) DM 40,-.* Akademie der Wissenschaften, 1957.
- [85] B. Sutton-Smith. *The Ambiguity of Play.* Harvard University Press, 2009.
- [86] M. Tardieu. *Le manichéisme. Que sais-je ?* Presses universitaires de France, 1981.
- [87] T. Taylor. *The philosophical and mathematical commentaries of Proclus ... on the first book of Euclid's Elements, and his life by Marinus, tr. with a prelim. dissertation on the Platonic doctrine of ideas by T. Taylor. (To which are added, A history of the restoration of the Platonic theology, by the latter Platonists: and a tr. of Proclus's Theological elements by T. Taylor).* The philosophical and mathematical commentaries of Proclus ... on the first book of Euclid's Elements, and his life by Marinus, tr. with a prelim. dissertation on the Platonic doctrine of ideas by T. Taylor. 1789.
- [88] Pliny the Elder. *Naturalis historia.*
- [89] Theocritus, R.L. Hunter, R.L. Hunter, P.E. Easterling, P. Hardie, and E.J. Kenney. *Theocritus: A selection: Idylls 1, 3, 4, 6, 7, 10, 11 and 13,* 1999.
- [90] J. Mark Thompson. *Defining the abstract. The Games Journal,* 2000.
- [91] H. von Baravalle. *Die Geometrie des Pentagramms und der Goldene Schnitt.* J.Ch. Mellinger, 1985.
- [92] Kurt Von Fritz. *The discovery of incommensurability by hippasus of metapontum. Annals of mathematics,* 46(2):242–264, 1945.

- [93] Helmuth von Glasenapp. *Die Philosophie der Inder*. 1949.
- [94] J.W. von Goethe. *Faust: Eine Tragödie. Erster Theil*. Hermann Passarge, 1841.
- [95] C. von Linné, M.S.J. Engel-Ledeboer, and H. Engel. *Carolus Linnaeus Systema Naturae, 1735: Facsimile of the 1st Ed.* Dutch classics on history of science. B. de Graff, 1964.
- [96] Heinrich Cornelius Agrippa von Nettesheim. *De occulta philosophia*. 1533.
- [97] L. von Von Mises. *Nationalökonomie: Theorie Des Handelns und Wirtschaftens*. Ludwig von Mises Institute, 1940.
- [98] E. Waldschmidt. *Die Stellung Jesu im Manichäismus*. Verlag der Akademie der Wissenschaften, 1926.
- [99] J. Webb. *The Flight from Reason. Age of the irrational*. Macdonald and Company, 1971.
- [100] Diederik Wentink. *Analysis and Implementation of the game Gïpf*. 2001. (PhD Thesis).
- [101] S. Wolfram. *A New Kind of Science*. WOLFRAM MEDIA Incorporated, 2018.
- [102] Ernst Zermelo. Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. In *Proceedings of the Fifth International Congress of Mathematicians*, pages 501–504, 1913.

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